Exam answers Asset Pricing Theory August 2011

Academic Aims Asset Pricing Theory

Modern finance theory analyses investment decisions under uncertainty with regard to the pricing of traded assets including derivatives. Throughout the course students will acquire an introductory understanding of the main elements in modern financial theory and the ability to apply the different models and techniques to problems of a limited complexity. Students are also expected to obtain an understanding of the mathematical methods related to the theories of finance including selected proofs and the associated practical methods.

The excellent performance is characterized by a good knowledge of the theories, methods, models and proofs covered in the course together with the ability to apply these competencies to explicit problems, in particular within:

Discrete time models

The one-period-model under uncertainty, equilibrium prices, the binomial model, arbitrage-pricing theory (ATP) and the multi-period-model.

Continuous times models

Wiener processes and Ito's lemma; Black-Scholes-Mertons model; martingales and measures; term structure models, models of the short rate, the Heath, Jarrow and Morton model (HJM); credit risk.

Problem 1 Market price of risk

(a) if we let the Π consist of $f_2\sigma_2$ of derivative 1 and $-f_1\sigma_1$ of derivative 2 then we got:

$$d(= \sigma_2 f_2 (\mu_1 f_1 dt + \sigma_1 f_1 dz_t) - \sigma_1 f_1 (\mu_2 f_2 dt + \sigma_2 f_2 dz_t) = f_1 f_2 (\sigma_2 \mu_1 - \sigma_1 \mu_2$$

Since the Wiener process is hedged out the portfolio is riskless.

(b) Since the above portfolio is riskless it must have the drift of the riskless interest rate r multiplied with the value of the portfolio:

$$f_1 f_2 (\sigma_2 \mu_1 - \sigma_1 \mu_2) = (r = (\sigma_2 f_1 f_2 - \sigma_1 f_2 f_1) r \square (\Leftrightarrow \tau) (\sigma_2 \mu_1 - \sigma_1 \mu_2) = \sigma_2 r - \sigma_1$$

(c) We reorganise the equation from (b) to:

$$(\mu_1 - r)/(_1 = (\mu_2 - r)/(_2 = ($$

and we define λ to be the ratio excess drift (drift minus riskless interest rate) divided by the volatility.

Since this ratio can be described from the parameters from the process of only one derivative and independent of the other derivative, the market price of risk will not depend on the nature of the derivatives.

If the investors are risk neutral then $\lambda = 0$, and $\mu = r$ for all f.

(d) From the market price of risk above we can see that $\mu = r + \lambda\sigma$ for all derivatives only depending on time and V, and by substituting that into the drift term of f, we can show what we is asked.

Problem 2 One factor models of the interest rate

(a)

$$dr_t = (b - ar_t)dt + \sigma\sqrt{r_t}dz_t$$

(b) the drift term of the SDE in both Vasiček and CIR ($b - ar_t$) are mean reverting processes, it can be rewritten as $a(b/a - r_t)$. This means if r_t is below b/a it will have a positive drift, and if it is above it will have a negative drift. b/a can be seen as the equilibrium level of the short rate. a is then the speed of which the rate returns towards the equilibrium.

This is a nice feature for interest rates, since they do follow stationary processes, and the above drift term does the same. I.e. it will not diverge.

(c) The volatility term of the Vasiček model is independent of r, and gives expected future short rates that are normal distributed, whereas the volatility of CIR is proportional to the square root of the rate. I.e. the volatility goes towards 0, when the rate goes towards 0, this feature secure that the short rate will not become negative in the CIR model, that is possible in Vasiček.

(d) The term structures are affine, means that the zero coupon yields y(t,T) can be written as an affine function of r_t, i.e. $Y(t,T) = A(T-t) + B(T-t)r_t$, giving zero coupon bond prices of the form $P(t,T) = \bar{A}(T-t)e^{-\bar{B}(T-t)r_t}$ where $\bar{A}(\tau) = e^{-A(\tau)\tau}$ and $\bar{B}(\tau) = B(\tau)\tau$ since

$$Y(t,T) = -\frac{\ln P(t,T)}{(T-t)}$$

(e) Since dr_t follows this mean reverting process, we need to change the process to say something about the mean, so we introduce a new variable $X_t = g(r_t, t) = e^{at}r_t$ and use Itô to find dg:

$$dg(r_t, t) = \left(ae^{at}r_t + e^{at}(b - ar_t)\right)dt + e^{at}\sigma dz_t = be^{at}dt + \sigma e^{at}dz_t$$

By taking the stochastic integral we can find X_t:

$$X_t = r_0 + \int_0^t b e^{as} ds + \int_0^t \sigma e^{as} dz_s = r_0 + \frac{b}{a}(e^{at} - 1) + \int_0^t \sigma e^{as} dz_s$$

We can now find an expression for the short rate:

$$r_t = X_t e^{-at} = e^{-at} r_0 + \frac{b}{a}(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dz_s$$

To find the mean, we know the last term has an expected mean of 0 and get:

$$E[r_t] = e^{-at} r_0 + \frac{b}{a}(1 - e^{-at})$$

(f) By setting $\alpha_t = e^{-at}$ and $r_\infty = \frac{b}{a}$ in the result above we get the wanted expression.

$$E_0[r_t] = \alpha_t r_0 + (1 - \alpha_t) r_\infty$$

So the expected value of r_t is a weighted average of the short term rate and the equilibrium rate in this model.

And since $\alpha_0 = 1$ and $\alpha_\infty = 0$ we have that the expectation of r_t goes towards r_0 for t going towards 0, and that the expectation of r_t goes towards r_∞ for t going towards ∞ .

(g) Given the Lemma stated in the problem and the SDE from (e) we can find the variance of r_t.

$$Var[r_t] = \sigma e^{-at} \int_0^t e^{as} dz_s = \sigma e^{-at} \frac{1}{a}(e^{at} - 1) = \frac{\sigma}{a}(1 - e^{-at})$$

Problem 3 FX securities and two factor Itô

(a) Since X is the USD / EUR and Y is USD/ GBP then V = X / Y = USD / EUR /(GBP / USD)
= GBP / EUR, i.e. the amount of GBP to pay one EUR.

(b) Taking the derivatives and putting into Itô's Lemma gives us:

$$\mu^Y Y_t dt + \sigma^{\square\uparrow}$$

The geometric Brownian motion we were asked to find.

(c) We know that z_t^X and z_t^Y are wiener processes, i.e. they satisfy the stated conditions.

1) $z_0^V = \alpha z_0^X - \beta z_0^Y = \alpha 0 + \beta 0 = 0$

2) Since both z_t^X and z_t^Y have independent increments, then will z_t^V also have independent increments.

3) Since both $z_t^X - z_s^X$ and $z_t^Y - z_s^Y$ both are Gaussian, then will $z_t^V - z_s^V$ also be Gaussian.

$$E[z_t^V - z_s^V] = E[\alpha z_t^X - \beta z_t^Y - (\alpha z_s^X - \beta z_s^Y)] = \alpha E[z_t^X - z_s^X] - \beta E[z_t^Y - z_s^Y] = \alpha 0 - \beta 0 = 0$$

$$Var[z_t^V - z_s^V] = Var[\alpha z_t^X - \beta z_t^Y - (\alpha z_s^X - \beta z_s^Y)] = Var[\alpha(z_t^X - z_s^X) - \beta(z_t^Y - z_s$$

So $z_t^V - z_s^V$ has the Gaussian distribution $N(0, \sqrt{t-s})$.

4) Since both z_t^X and z_t^Y both are continuous, then will $z_t^V - z_s^V$ also be continuous.

(d) The wiener process giving the stochastic to the currency V is a linear combination of the processes to X and Y and we have $z_t^V = \alpha z_t^X + \beta z_t^Y$ where $\alpha^2 + \beta^2 - 2\rho\,\alpha\beta = 1$.

Setting $\alpha = \frac{\sigma^X}{\sigma^Z}$ and $\beta = \frac{\sigma^Y}{\sigma^Z}$ we get $\left(\frac{\sigma^X}{\sigma^Z}\right)^2 + \left(\frac{\sigma^Y}{\sigma^Z}\right)^2 - 2\rho\,\frac{\sigma^X}{\sigma^Z}\frac{\sigma^Y}{\sigma^Z} = 1$ $\Leftrightarrow$

$$(\sigma^X)^2 + (\sigma^Y)^2 - 2\rho\,\sigma^X\sigma^Y = (\sigma^Z)^2 \Leftrightarrow \rho = \frac{(\sigma^X)^2 + (\sigma^Y)^2 - (\sigma^Z)^2}{2\sigma^X\sigma^Y}$$

(e) Since the correlation between the two FX-rates depends on the volatility of the three FX-rates, I know that the correlation will change if the FX-volatilities change. So if I am long correlation risk, i.e. I gain money if correlation goes up. That gives me a gain if the volatility of EURUSD or GBPUSD goes up (σ^X and σ^Y) or if GBPEUR goes down σ^Z.